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LETTER TO THE EDITOR

Ground state of 1D bosons with delta interaction: link to the BCS model

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Abstract
The Bethe roots describing the ground-state energy of the integrable one-dimensional (1D) model of interacting bosons with weakly repulsive two-body delta interactions are seen to satisfy the set of Richardson equations appearing in the strong coupling limit of an integrable BCS pairing model. The BCS model describes boson–boson interactions with zero centre of mass momentum of pairs. It follows that the Bethe roots of the weakly interacting boson model are given by the zeros of Laguerre polynomials. The ground-state energy and the lowest excitation are obtained explicitly via the Bethe roots. A direct link has thus been established, in the context of integrable 1D models, between bosons interacting via weakly repulsive two-body delta interactions and strongly interacting BCS boson pairs.

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There has been a revival of interest in the exactly solved 1D model of interacting bosons [1]. The conventional description of the model is N interacting bosons governed by the Hamiltonian

$$\mathcal{H} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1\leqslant i<j\leqslant N} \delta(x_i - x_j) \tag{1}$$

and constrained by periodic boundary conditions to a line of length L. Here $2c$ is the strength of the two-body delta interaction, with $\hbar = 2m = 1$. The wavefunctions are given in terms of the Bethe ansatz by

$$\psi(x_1, \ldots, x_N) = \sum_{p} A(p) \exp\left(\mathrm{i} \sum_{j} k_{p_j} x_j\right) \tag{2}$$

in the region $x_1 < x_2 < x_3 < \cdots < x_N$. The summation extends over all permutations p of momenta $\{k_j\}$ and $A(p)$ are coefficients depending on p. The eigenvalues are given by $\mathcal{E} = \sum_{j=1}^{N} k_j^2$ where $\{k_j\}$ satisfy the Bethe equations

$$\exp(\mathrm{i}k_j L) = -\prod_{\ell=1}^{N} \frac{k_j - k_\ell + \mathrm{i}c}{k_j - k_\ell - \mathrm{i}c} \qquad \text{for} \quad j = 1, \ldots, N. \tag{3}$$

The Bethe roots k_j are known to be real for repulsive interactions, $c > 0$, for which the ground-state energy and excitation spectrum have been studied extensively[4].

The revival of interest in this model has been inspired by recent experimental and theoretical work on low-dimensional trapped boson gases at ultracold temperatures [8–15], particularly with regard to highly elongated traps which give possible realizations of a 1D quantum gas as well as Bose–Einstein condensates. In a remarkable recent experiment, a 1D quantum gas was created in an optical lattice in which the interactions between trapped ultracold Rb atoms were modified to bring about a continuous passage from the weakly interacting regime to the strongly correlated Tonks–Girardeau gas (see [8] and references therein). The key parameter is essentially the ratio γ of interaction to kinetic energy. In this way, the pronounced fermionic behaviour of the Tonks–Girardeau gas was observed in the strong coupling regime [8].

Here we consider asymptotic solutions to the Bethe equations of the interacting boson model in the strong and weak coupling limits. We see that the momentum density distribution agrees with the observed profiles of the 1D interacting quantum gas [8]. Moreover, we find a link between the Bethe equations (3) in the weak coupling regime and the Richardson equations [16] for the BCS boson pairing model in the strong coupling regime [17–19]. The BCS model describes boson–boson interactions with zero centre of mass momentum of pairs. In this way, the Bethe roots for the ground-state energy of the weakly interacting Bose gas are characterized by either the roots of Hermite polynomials or appropriate Laguerre polynomials. In the other direction, we thus see an explicit connection between strongly interacting Cooper pairs described by the BCS boson pairing model and bosons interacting through weakly repulsive two-body delta interactions. We begin by looking at the strong coupling limit.

Strong coupling limit. In the $Lc = \infty$ limit, for the dilute gas ($L > N$) there is a well-known connection to non-interacting fermions, known as the Tonks–Girardeau gas [20]. The Bethe equations (3) reduce to $\exp(\mathrm{i}k_j L) = (-1)^{N-1}$, with solutions $2\ell\pi/L$ with ℓ integer for odd N and half-odd-integer for even N. Corrections to order $1/c$ can be readily obtained from the asymptotic solutions of the Bethe equations (3). Define $\lambda = Lc/N$. The roots are real and symmetric about the origin, with $\left\{\pm k_{2m-1}, m = 1, \ldots, \frac{N}{2}\right\}$ for even N, where $k_\ell = \frac{\ell\pi}{L}\left(1 - \frac{2}{\lambda}\right)$. For odd N, the roots are zero and $\left\{\pm k_{2m}, m = 1, \ldots, \frac{N-1}{2}\right\}$. The ground-state energy follows directly from these asymptotic solutions, with

$$\frac{\mathcal{E}_0}{N} = \frac{1}{3}(N^2 - 1)\frac{\pi^2}{L^2}\left(1 - \frac{2N}{Lc}\right)^2 \approx \frac{\pi^2\rho^2}{3}\left(1 - \frac{4}{\lambda}\right), \tag{4}$$

where $\rho = N/L$. This result coincides with that of the perturbation theory approach (see, e.g., [21]). At zero temperature, the chemical potential is given by $\mu = \pi^2\rho^2 - 16\pi^2\rho^3/3c$. The two-body correlation function $g_2 = 4\pi^2\rho_0^2/\gamma^2$ follows from the ground-state energy and the Hellmann–Feynman theorem, with g_2 giving the rates of two-body inelastic processes in 1D trapped boson gases [13]. Moreover, we see that the momentum density distribution for the 1D N-body interacting bosons is flat due to the symmetric equal-spacing distribution in

[4] See, e.g., [1–7] and references therein.

momentum space. This qualitatively coincides with the experimental momentum profile of the Tonks–Girardeau gas in the strong coupling limit, for example, $\gamma \approx 204.5$ [8]. In this regime, the strong repulsive delta potential prevents the bosons from occupying the same position. Therefore, they are spread out and extend to a larger region in momentum space than in the case of weakly interacting bosons.

Weak coupling limit. The situation in the weak coupling limit is far more interesting. The ground state is an analytic function for $c > 0$. Considering the limit $Lc \ll 1$ and after some tedious case-by-case calculations for $N = 2, 3, \ldots, 12$, we find that the Bethe roots satisfy the set

$$k_j = \frac{2\pi d_j}{L} + \frac{2c}{L}\sum_{\ell=1}^{N'} \frac{1}{k_j - k_\ell}, \qquad j = 1, \ldots, N, \tag{5}$$

of nonlinear algebraic equations. Here the summation excludes $j = \ell$ and $d_j = 0, \pm 1, \pm 2, \ldots$ denotes excited states for fixed N. The ground state has zero total momentum, with $d_j = 0$ for $j = 1, \ldots, N$. We have compared numerical solutions of equations (5) with the numerical solutions of the Bethe equations (3) for small c. Indeed, the quantum number d_j characterizes all states. The asymptotic equations (5) are closely related to Stieltjes problems [22]. The ground-state energy per particle

$$\frac{\mathcal{E}_0}{N} = \frac{c(N-1)}{L}, \tag{6}$$

follows directly from (5). The nonlinear algebraic equations (5) for the ground state were found by Gaudin [5], who showed that the k_j are roots of Hermite polynomials of degree N, namely $H_N(k) = 0$.

We are also interested in the lowest excited state, which has total momentum $k = 2\pi/L$. Without loss of generality, we may take $d_1 = 1, d_2 = \cdots = d_N = 0$. The lowest excitation energy per particle

$$\frac{\mathcal{E}_1}{N} \approx \frac{c(N-1)}{L} + \frac{2\pi}{LN}\left(\frac{2\pi}{L} + \frac{2c(N-1)}{L} + \frac{c^2(N-2)^2}{L2\pi}\right) \tag{7}$$

follows from solving equations (5). Correspondingly, the largest Bethe root is

$$k_1 = \frac{2\pi}{L} + \frac{2c(N-1)}{L} + \frac{c^2(N-2)^2}{L2\pi}.$$

It is clearly seen that the energy gap vanishes in the thermodynamic limit. Some numerical exploration reveals that the excitation energy per particle (7) is very accurate for arbitrary number of bosons. Other excitation energies depend on the assignments of d_j, with total momentum $k = \sum_j^N 2d_j\pi/L$.

The connection between the Bethe roots for the ground-state energy and the roots of $H_N(k)$ provides a systematic way for studying quantities such as correlations and the momentum distribution function. The normalized momentum density distribution is given by the semi-circle law [5]

$$n(k) = \frac{L}{2N\pi c}(4c\rho - k^2)^{\frac{1}{2}}. \tag{8}$$

We show the momentum density distribution in figure 1 obtained by numerically solving the Hermite polynomials of degree $N = 60$ and 100 for different values of c. The results fit very well the analytical expression (8). A similar semi-circular momentum density distribution has been found in the ferromagnetic Heisenberg chain [23]. We see that the distribution at small

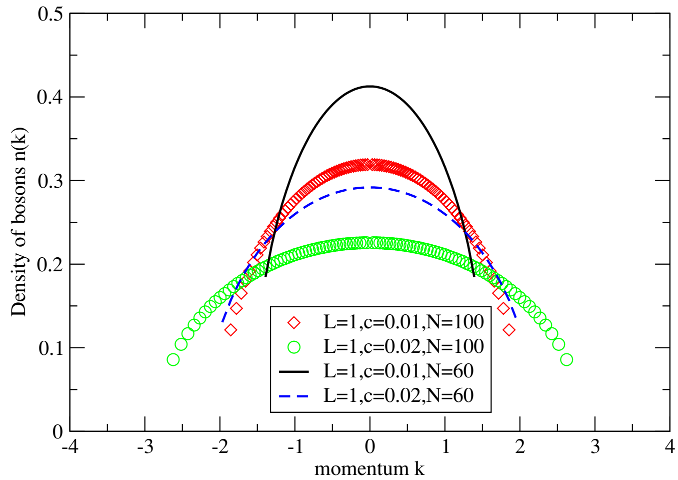

Figure 1. The density $n(k)$ of momentum distribution (normalized) for $N = 60$ and 100 particles: the density distribution function sensitively depends on interaction strength, particle number and the length L for the 1D boson gas with repulsive delta interaction in the limit $Lc \ll 1$.

(This figure is in colour only in the electronic version)

momentum is rather flat, with an almost linearly decreasing region at large momentum. The stronger the interaction strength, the larger the momentum distribution region. This reveals a significant signature of the 1D boson gas in the weakly repulsive limit $Lc \ll 1$. Remarkably, this behaviour was recently observed in the experiment for weakly interacting bosons with $\gamma \approx 0.5$ [8]. If $c = 0$ all the particles condense in the ground state at zero temperature.

In order to link the 1D boson gas with two-body delta interactions to the BCS type of pairing models, we consider first an even number of bosons, with $N = 2M$. We also find, to first order in c, that the Bethe roots for the ground state are of the form

$$k_{1,2} = \pm\sqrt{E_1}, k_{3,4} = \pm\sqrt{E_2}, \ldots, k_{2M-1,2M} = \pm\sqrt{E_M}, \tag{9}$$

where the E_i satisfy the equations

$$-\frac{L}{2c} + \sum_{j=1}^{M'} \frac{2}{E_i - E_j} = -\frac{1}{2E_i}, \tag{10}$$

for $i = 1, \ldots, M$, where the summation excludes $j = i$.

Now similar equations have arisen in a number of contexts [17, 22]. Of particular interest here is the connection between equation (10) and Richardson's equations for the BCS pairing model [24, 25] in the strong coupling limit [17]. However, the precise link between the integrable boson model with weakly repulsive delta interaction and the standard BCS model with strong attractive pairing interaction is quite subtle. In terms of the Cooper pair energies E_i participating in the scattering process, these latter equations are

$$-\frac{1}{\lambda_{\mathrm{BCS}} d} + \sum_{j=1}^{r'} \frac{2}{E_i - E_j} = \frac{n'}{E_i}, \tag{11}$$

for $i = 1, \ldots, r$. Here $n' = n - 2(m - r)$, where n is the number of unblocked energy levels, m is the number of Cooper pairs and r is the number of nonvanishing roots to order λ_{BCS}, where λ_{BCS} is the superconducting coupling constant, with d the mean level spacing.

In the BCS model, the parameters $\{E_i\}$ in equation (11) represent the energies of r Cooper pairs participating in the scattering. Now the ground state of the BCS model in the strong coupling limit has $r = m$ $(n' = n)$, the first degenerate group of excited states corresponds to $r = m - 1$, etc [17]. Direct comparison between equations (10) and (11) gives $r = M$ and $n' = -\frac{1}{2}$, leading to the ground state where the total number of Cooper pairs is $m = r = M$. However, in the strong coupling limit, all of the energy levels collapse into one multiply-degenerate level, namely $n = 1$.

Obviously, this state appears to lie outside the physical regime of the BCS model [17] due to the presence of the negative fractional quantum number $n' = -\frac{1}{2}$. The main reason for this is that in the standard BCS pairing interaction, the electrons are paired into singlets rather than into triplets with zero centre of mass momentum of pairs. Therefore the pairing interaction term contains only a pair of attractive electrons with opposite spin and momenta so that the degeneracy at each energy state is a doublet, with level degeneracy $\Omega = 2$ [24]. Moreover, the commutation relation between BCS pairs (hard-core bosons) is not the same as that for bosons.

Before addressing the correspondence, we can nevertheless make use of the observation [17] that the nonvanishing roots of the BCS equation (11) are given by the zeros of an associated Laguerre polynomial. It follows that the Bethe roots of the weakly interacting boson gas in equation (10) are also given in terms of the associated Laguerre polynomial $L_n^k(x)$ by

$$L_M^{-\frac{1}{2}}\left(\frac{LE_i}{2c}\right) = 0, \tag{12}$$

for $i = 1, \ldots, M$, where

$$L_n^k(x) = \sum_{m=0}^{n} \frac{(n+k)!}{(n-m)!(k+m)!m!} x^m. \tag{13}$$

Explicitly, for $M = 1$ $(N = 2)$, $E_1 = c/L$ and for $M = 2$ $(N = 4)$, $E_1 = c(3-\sqrt{6})/L$, $E_2 = c(3+\sqrt{6})/L$.

For an odd number of bosons, $N = 2M + 1$, we find the ground-state Bethe roots to be given by

$$k_1 = 0, k_{2,3} = \pm\sqrt{E_1}, k_{4,5} = \pm\sqrt{E_2}, \ldots, k_{2M,2M+1} = \pm\sqrt{E_M}, \tag{14}$$

where the E_i satisfy the equations

$$\sum_{j=1}^{M'} \frac{2}{E_i - E_j} - \frac{L}{2c} = -\frac{3}{2E_i}, \tag{15}$$

for $i = 1, \ldots, M$. Comparing again with equation (11), $n' = -\frac{3}{2}$ again with $r = m = M$. In this case the Bethe roots are given by

$$L_M^{\frac{1}{2}}\left(\frac{LE_i}{2c}\right) = 0, \tag{16}$$

for $i = 1, \ldots, M$. The first few cases are $E_1 = 3c/L$ for $M = 1$ $(N = 3)$ with $E_1 = 3c/L$ and $E_1 = c(5-\sqrt{10})/L$, $E_2 = c(5+\sqrt{10})/L$ for $M = 2$ $(N = 5)$.

For both the even and odd cases our results indicate that the Bethe roots approach the origin proportional to $\sqrt{c}$ as $c \to 0$ for fixed N and L. We have checked this directly via numerical solution of the Bethe equations (3).

We now address the precise correspondence between the boson and BCS models. In view of the ground-state energy $\mathcal{E}_0 = \sum_{j=1}^{N} k_j^2$ of the boson model, we can map two bosons with opposite momenta $\pm k_j$ onto one Cooper pair of bosons with energy $E_j = 2k_j^2$. In such a way, we can show that the ground state of the boson model with weakly repulsive delta interactions can be described by the BCS boson pairing model with strong pairing interaction. In particular, we need to consider the boson pairing model [18, 26]

$$\mathcal{H} = \sum_{\ell} \epsilon_\ell \hat{n}_\ell + 2g \sum_{j,j'=1}^{n} A_j^\dagger A_{j'}, \tag{17}$$

based on the $su(1,1)$ algebra. Here $\hat{n}_\ell = \frac{1}{2}a_\ell^\dagger a_\ell + \frac{1}{4}$ denotes the particle number operator at level ℓ and $A_j^\dagger = \frac{1}{2}a_j^\dagger a_j^\dagger$ creates a boson pair with zero angular momenta at the level j. The single-particle operators $a^\dagger$ and a satisfy Bose commutation relations. Essentially, the single-particle state can be its own time reversal state for bosons at same levels. Therefore at every energy level there is a single pair with degeneracy $\Omega = 1$ [18]. For the boson pairing model, the pair energies are real. In the above equation, n is the number of unblocked energy levels and g is the pairing interaction. The energy of the pairing model (17) is given by

$$\mathcal{E}_{\rm BCS} = \sum_{\ell} \epsilon_\ell \nu_\ell + \sum_{j=1}^{m} E_j, \tag{18}$$

where the pair energies E_j satisfy Richardson equations of the form [18]

$$-\frac{1}{2g} - \sum_{\ell=1}^{n} \frac{2d_\ell}{2\epsilon_\ell - E_k} + \sum_{i=1}^{m'} \frac{2}{E_k - E_i} = 0. \tag{19}$$

Here m denotes the total number of boson pairs and $d_\ell = (\nu_\ell/2) + (\Omega_\ell/4)$ is the effective pair degeneracy of single-particle level ℓ, where ν_ℓ denotes the number of unpaired particles in level ℓ. Linking to the boson pairing model [18], here the degeneracy $\Omega_\ell = 1$ and $\nu_\ell = 0$.

In terms of the Cooper pair energies E_i participating in the scattering process in the strong coupling limit, equations (19) are

$$-\frac{1}{2g} + \sum_{j=1}^{r'} \frac{2}{E_i - E_j} = -\frac{n'}{E_i}, \tag{20}$$

for $i = 1, \ldots, r$. Here $n' = \frac{n}{2} + 2(m - r)$, where n is the number of energy levels, m is the number of Cooper pairs and r is the number of nonvanishing roots to order g. Direct comparison between equations (10) and (20) gives $g = c/L$, $r = m = M$ and $n = 1$, leading to a state of the BCS boson pairing model (17) with energy

$$\mathcal{E}_{\rm BCS} = \sum_{i=1}^{r} E_i = 2gr(n' + r - 1). \tag{21}$$

In the strong coupling limit, all energy levels can collapse into one multiply-degenerate level. It is thus reasonable that $n \approx 1$. Indeed, this state appears to lie in the physical regime of the boson pairing model (17). The ground state of the weakly interacting high-density boson model thus corresponds to a state of the strongly coupled BCS boson pairing model where the level spacing is ignored in comparison with the scattering pairing energy. From equations (10) and (15), the energy per particle of the weakly interacting Bose gas follows as

$$\frac{\mathcal{E}_0}{N} = \sum_{i=1}^{N} \frac{k_i^2}{N} = \frac{2\mathcal{E}_{\rm BCS}}{N} = \frac{c(N-1)}{L}. \tag{22}$$

This agrees with equation (6) and the results of [1] (see also [27, 28]). We see that it also coincides with the ground-state energy per particle of the strongly coupled BCS boson pairing model (17).

Further, if the Cooper pair is defined [29] as $b_{\mathbf{kK}} = c_{k_2\downarrow}c_{k_1\uparrow}$, $b^{\dagger}_{\mathbf{kK}} = c^{\dagger}_{k_1\uparrow}c^{\dagger}_{k_2\downarrow}$ where $\mathbf{k} \equiv \frac{1}{2}(k_1 - K_2)$ and $\mathbf{K} \equiv (k_1 + K_2)$, it is shown that $\left[b_{\mathbf{kK}}, b^{\dagger}_{\mathbf{k'K}}\right] = 0$ for $\mathbf{k} \neq \mathbf{k}'$, while $[b_{\mathbf{kK}}, b_{\mathbf{k'K}}] = \left[b^{\dagger}_{\mathbf{kK}}, b^{\dagger}_{\mathbf{k'K}}\right] = 0$ always hold. The single operators $c^{\dagger}$ and c satisfy Fermi anticommutation relations. If the relative momentum vector of paired electrons lies inside the overlap of the two spherical shells in momentum space, the Cooper pair with nonzero centre of mass momentum, i.e. $\mathbf{K} \equiv (k_1 + K_2)$, is possible and physical [29]. This means that the Cooper pairs can occupy a state of energy $\epsilon_{\mathbf{K}}$ with same $\mathbf{K}$ but different values of $\mathbf{k}$. Thus such Cooper pairs act as bosons and can collapse into a Bose condensate. In this BCS pairing model, the degeneracy of single-particle level Ω_ℓ would not be restricted to be even such that the quantum number n' in equation (11) can be fractional and negative. In this case, the single-particle states for a Cooper pair are not time-reversal eigenstates. For the standard BCS model, $\mathbf{K} = 0$, such that the state $\mathbf{k}\uparrow$ is occupied so is the state $-\mathbf{k}\downarrow$. Therefore we can expect that the BCS model of fermion pairs with nonzero centre-mass momentum would exhibit some of the behaviour of the 1D boson gas with two-body delta interactions.

In conclusion, we have looked in detail at the many-body solution of the Bethe equations for the interacting boson model in the strongly and weakly repulsive coupling regimes. We found a remarkable connection with the Richardson equations for the strong coupling limit of the boson pairing model, suggesting a direct link between strongly interacting Cooper pairs and weakly repulsive bosons. This link deserves further investigation, particularly in light of the revival of interest in the interacting Bose gas and the BEC/BCS crossover [30, 31].

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